Lecture 2 - Length Contraction

A Puzzle

We are all aware that if you jump to the right, your reflection in the mirror will jump left. But if you raise your hand up, your reflection will also raise its hand up.

Why does a mirror only flip left-right *and not up-down*? How does it "know" which direction to flip?

(1)

Solution

A mirror doesn't actually flip left-right, but rather in the forward-backward direction. To get your mirror image, you need to flip your body forward and backwards (i.e. pull your head out in front of your nose (please don't try that...)). Because we are left-right symmetric, we see our reflection as our normal self but with its left and right swapped. This YouTube video also examines why words appear to be flipped left-right in the mirror.

Recap of Time Dilation

- Two events that are simultaneous in one frame are not necessarily simultaneous in another frame. The only exception is that if two events occur at the same time and location in one frame, then they will be simultaneous in all reference frames.
- If two objects move past each other, they both see each other's clocks moving slowly (by the factor $\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$) in their own frame. That is fine, because time means different things in the different reference frames.

The Head Start

Example

Two clocks are positioned at the ends of a train of length L (as measured in its own frame). They are synchronized in the train frame. The train travels past a person at speed v. It turns out that if the person observes the clocks at simultaneous times in her frame, she will see the rear clock showing a higher reading than the front clock. By how much?

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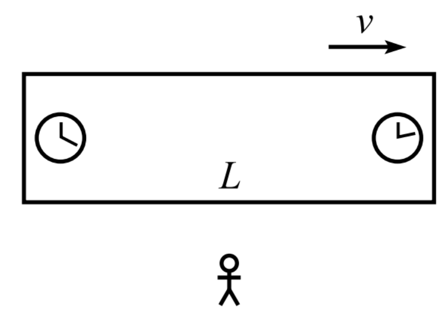

<u>Solution</u>

Let’s focus on the two pieces of information we have about the clocks:

- The clock’s are synchronized in the train’s frame
- The clocks are simultaneously observed in the person’s frame

Because these two events occur in different frames, we need a way to transfer one of these pieces of information into the other frame.

Here’s the game plan: Let’s put a light source on the train, positioned so that the light hits the clocks at the ends of the train at the same time *in the person’s frame*. We will say that the person reads the two clocks at exactly the instant when the two photons hit their respective clocks. Then we can translate this setup into the train’s frame where we know that the two clocks are synchronized, so that we measure the time difference between when the two photons hit the two ends of the trains.

Recall from last time that the relative speed (as viewed in the person’s frame) between the photon shooting backwards and the rear of the train is $c + v$, while the relative speed between the photon moving forward and the front of the train is $c - v$. So if we put the light source in the middle of the train, it will hit the back of the train before the front of the train (in the person’s frame). We don’t want that - instead, we are trying to place the light source so that it will hit both ends of the train simultaneously.

To ensure that the two photons hit both ends of the train simultaneously (in the person’s frame), we should position the light source a distance $\frac{L(c+v)}{2c}$ from the rear of the train and $\frac{L(c-v)}{2c}$ from the front of the train as follows (but see the *Technical Note* below):

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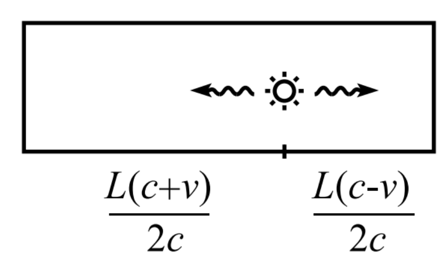

The time when the person will observe each clock is now explicitly defined, so let us consider this situation from the train’s frame. In the train’s frame, the light must travel an extra distance of $\frac{L(c+v)}{2c} - \frac{L(c-v)}{2c} = \frac{L v}{c}$ to reach the rear clock. The extra time it takes for light to reach this rear clock is therefore $\frac{L v}{c^2}$. Hence, the rear clock reads $\frac{L v}{c^2}$ more when it is hit by the backward photon, and this leads to the difference in times that the person on the ground will see:

$$\text{Difference in reading} = \frac{L\,v}{c^2} \tag{2}$$

Some interesting points about this result:

- The length L that appears here is the length of the train in its own frame, and not the shortened length that you observe in your frame; we will learn about this length contraction below.
- The rear clock is not ticking at a faster rate than the front clock. They run at the same rate (since both have the same time-dilation factor relative to the person). The back clock is simply a fixed time ahead of the front clock, as seen by the person.
- To the person outside the train, it appears as though the clocks are not synchronized, but rather that the rear clock has a constant offset in time. What would the situation look like to a person standing still on the train? That person would tell the person standing on the ground outside the train, "The clocks are synchronized, but you did not read them at the same time. First you read the clock at the front of the train, then you waited a time $\frac{L\,v}{c^2}$, and then you read the rear clock." This gives you a sense of how events in two different reference frames are interconnected.

Technical Note: In the person's frame, the train will actually be shortened due to length contraction (which we will discuss next in this lecture). So the light source should be positioned a distance $\frac{L'(c+v)}{2c}$ from the rear of the train and $\frac{L'(c-v)}{2c}$ from the front of the train, where $L' = \frac{L}{\gamma}$. But when we then begin to analyze the setup from the train's frame (where the train once again has length L), then the partitioning of the train shown above is exactly correct. Thus, glossing over this point does not change the result. □

Length Contraction

Complementary Section: A Simple Proof

Let B stand on the ground, next to a stick of length l. Let A fly past the stick at speed v. In B's frame, it takes A a time of $\frac{l}{v}$ to traverse the length of the stick. But given the effects of time dilation, a watch on A's wrist will advance by a time of only $\frac{l}{\gamma\,v}$ while he traverses the length of the stick.

How does A view the situation? He sees the ground and the stick fly by with speed v. As stated above, the time between the two ends passing him is $\frac{l}{\gamma\,v}$ (as per his time dilated watch). To get the length of the stick in his frame, he simply multiplies the speed times the time. That is, he measures the length to be $\left(\frac{l}{\gamma\,v}\right) v = \frac{l}{\gamma}$, so that the stick appears to be contracted. This argument shows how time and space are tied together.

As with time dilation, this length contraction is a bit strange, but there doesn't seem to be anything actually paradoxical about it, until we look at things from A's point of view. To make a nice symmetrical situation, suppose A and B are both standing on identical trains, but B's train is motionless with respect to the ground while A's train is moving with velocity v. As discussed above, A will see B's train as contracted. But in A's frame, B is flying by at speed v in the other direction. Neither train is any more fundamental than the other, so the same reasoning applies, and B sees A's train as length contracted by the same amount as A sees B's train is contracted.

But how can this be? Are we claiming that A's train is shorter than B's, and also that B's train is shorter than A's? Does the situation look like this

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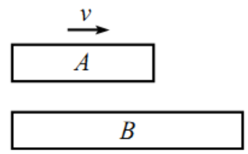

or does it look like this

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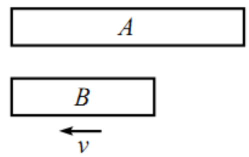

Well...it depends. The word "is" in the above paragraph is a very bad word to use, and is generally the cause of all the confusion. There is no such thing as "is-ness" when it comes to lengths. It makes no sense to say what the length of the train really is. It only makes sense to say what the length is in a given frame. The situation doesn’t really look like one thing in particular. The look depends on the frame in which the looking is being done.

Let’s be a little more specific. How do you measure a length? You write down the coordinates of the ends of something measured simultaneously, and then you take the difference. But the word "simultaneous" here should send up all sorts of red flags. Simultaneous events in one frame are not simultaneous events in another.

Stated more precisely, here is what we are really claiming: Let B write down simultaneous coordinates of the ends of A’s train, and also simultaneous coordinates of the ends of her own train. Then the difference between the former is smaller than the difference between the latter. Likewise, let A write down simultaneous coordinates of the ends of B’s train, and also simultaneous coordinates of the ends of his own train. Then the difference between the former is smaller than the difference between the latter. There is no contradiction here, because the times at which A and B are writing down the coordinates don’t have much to do with each other, due to the loss of simultaneity. As with time dilation, we are comparing apples and oranges.

A Complex but Very Satisfying Proof

Person A stands on a train which he measures to have length l' , and person B stands on the ground. A light source is located at the back of the train, and a mirror is located at the front. The train moves at speed v with respect to the ground. The source emits a flash of light which heads to the mirror, bounces off, then heads back to the source. By looking at how long this process takes in the two reference frames, we can determine the length of the train, as viewed by B.

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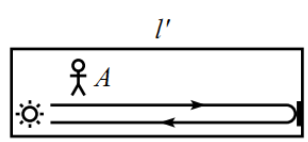

A's frame

In A’s frame, the round-trip time for the light is simply

$$t_A = \frac{2\,l'}{c} \tag{3}$$

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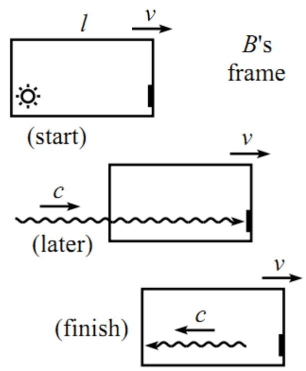

Things are a little more complicated in B's frame. Let the length of the train, as viewed by B, be l. For all we know at this point, l may equal l' , but we will soon find that it does not. The relative speed of the light and the mirror during the first part of the trip is $c - v$. The relative speed during the second part is $c + v$. During each part, the light must close a gap with initial length l. Therefore, the total round-trip time is

$$t_B = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2\,l\,c}{c^2-v^2} = \frac{2\,l}{c}\,\gamma^2 \qquad (4)$$

where you will recall from last time that

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \qquad (5)$$

But we know from time dilation that

$$t_B = \gamma\, t_A \qquad (6)$$

(You may be asking, why isn't the formula $t_A = \gamma\, t_B$. After all, we could view the observer in B to be moving relative to the stationary train in the train's reference frame. Very true, but completely irrelevant. For as discussed in Lecture 1 in the section "Understanding Time Dilation", the time dilation formula $t_B = \gamma\, t_A$ only holds if two events *occurred in the same location* in A's frame. This is certainly true, since the light beam began and ended up in exactly the same location in A's frame. Can the same thing be said in B's frame? No, because as seen in the figure above, the light beam started and ended at a different location; therefore *we could not* use the time dilation formula $t_A = \gamma\, t_B$ (I write it in toxic orange to stress that it is an **incorrect** formula in this situation))

Substituting from above, we find

$$l = \frac{l'}{\gamma} \qquad (7)$$

Since $\gamma \geq 1$, B measures the train to be shorter than (or, when $v = 0$, equal to) what A measures.

The term *proper length* is used to describe the length of an object in its rest frame. So l' is the proper length of the above train.

Notice that the length contraction factor γ is independent of position. That is, all parts of the train are contracted by the same amount. To prove this, we could put a number of small replicas of the above source-mirror system along the length of the train. They would all produce the same value for γ, independent of the position on the train.

Time Dilation vs Length Contraction

The *proper lifetime* of a particle is its lifetime as measured its own rest frame.

Example

Elementary particles called *muons* (which are identical to electrons, except that they are about 200 times as massive) are created in the upper atmosphere when cosmic rays collide with air molecules. The muons have an average lifetime of about 2×10^{-6} seconds (then they decay into electrons, neutrinos, and the like), and move at nearly the speed of light.

Assume for simplicity that a certain muon is created at a height of 50 km, moves straight downward, has a speed $v = 0.99998\, c$, decays in exactly $T = 2 \times 10^{-6}\, s$, and doesn't collide with anything on the way down.* Will the muon reach the Earth before it decays?

* Note: In the real world, the muons are created at various heights, move in different directions, have different speeds, decay in lifetimes that vary according to a standard half-life formula, and may very well bump into air molecules. So technically we've got everything wrong here. But that's no matter. This example will work just fine for the present purpose.

Solution

In the Earth's frame: In the Earth's frame, time dilation kicks in and the lifetime of the muon is extended by a factor of

$$\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \approx 160 \qquad (8)$$

The distance traveled in the Earth's frame is therefore $v\,(\gamma\, T) \approx 100$ km. Hence, the muon travels the 50 km, with room to spare.

In the muon's frame: In the muon's rest frame, the Earth is moving towards it with speed v, and the starting distance of the Earth is length-contracted to $\frac{50\text{ km}}{\gamma} \approx 300\, m$ (think of the 50 km column of air between the Earth and the muon as a train; in the muon's rest frame, this "train" of air is moving and hence is length-contracted). Since the Earth can travel a distance of 600 m during the muon's lifetime, the Earth collides with the muon, with time to spare.

Note that we are writing γ in all of these problems, but in reality every object in every reference frame has its own γ calculated based upon its own speed. For example, in the Earth's frame, $v_{\text{Earth}} = 0$ so that $\gamma_{\text{Earth}} = 1$ while $v_{\text{muon}} = v$ so that $\gamma_{\text{muon}} = \frac{1}{\left(1-\frac{v_{\text{muon}}^2}{c^2}\right)^{1/2}}$. In the muon's reference frame, $v_{\text{muon}} = 0$ so that $\gamma_{\text{muon}} = 1$ while $v_{\text{Earth}} = v$ so that $\gamma_{\text{Earth}} = \frac{1}{\left(1-\frac{v_{\text{Earth}}^2}{c^2}\right)^{1/2}}$. In this problem, because $\gamma_{\text{muon}} = \gamma_{\text{Earth}}$, we just wrote both of them as γ. But in later problems where multiple objects are moving at different speeds, each object's has its own frame-dependent γ.

The real-life fact that we actually do detect muons reaching the surface of the Earth in the predicted abundances (which would not happen without this special relativity effect) is one of the many experimental tests that support the relativity theory. □

Transverse Length Contraction

Example

Two meter sticks, A and B, move past each other. Stick A has paint brushes at its ends. Use this setup to show that in the frame of one stick, the other stick still has a length of one meter.

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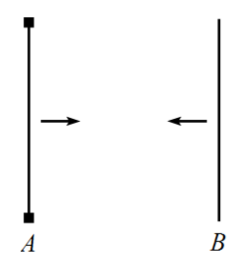

Solution

Assume for the sake of a contradiction that in A's frame, stick B is shortened.

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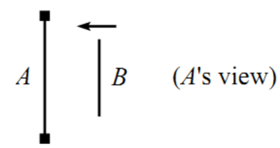

Then B won't reach out to the ends of A, so there will be no marks on B. But in this case, B must also see A shortened, so there will be marks on B. This is a contradiction.

Similarly, if we assume that in A's frame, stick B is lengthened, we also reach a contradiction.

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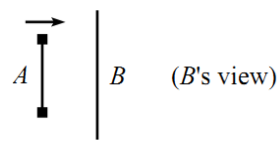

(Alternatively, we could have just used the ground reference frame where both sticks would be shrunk by the same factor, so the two ends of B would be painted; this B to retain its same length in A's reference frame, thereby implying no transverse length contraction.) □

Problems

Two Length Contractions

Example

Two trains, A and B, each have proper length L and move in the same direction. A's speed is $\frac{4}{5}c$, and B's speed is $\frac{3}{5}c$. A starts behind B.

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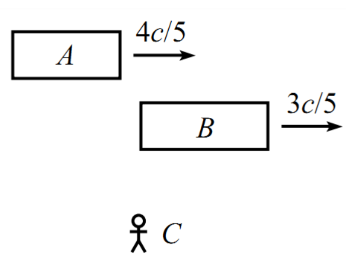

How long, as viewed by person C on the ground, does it take for A to overtake B? By this we mean the time between the front of A passing the back of B, and the back of A passing the front of B.

Solution

Relative to C on the ground, the γ factors associated with A and B are $\frac{5}{3}$ and $\frac{5}{4}$, respectively. Therefore, their lengths in the ground frame are $\frac{3}{5}L$ and $\frac{4}{5}L$. While overtaking B, A must travel farther than B, by an excess distance equal to the sum of the lengths of the trains, which is $\frac{4}{5}L + \frac{3}{5}L = \frac{7}{5}L$. The relative speed of the two trains (as viewed by C on the ground) is the difference of the speeds, which is $\frac{c}{5}$. The total time is therefore

$$t_C = \frac{7\,L/5}{c/5} = \frac{7\,L}{c} \qquad (9)$$

for A to overtake B. □

Explaining Length Contraction

Example

Two bombs lie on a train platform, a distance L apart. As a train passes by at speed v, the bombs explode simultaneously (in the platform frame) and leave marks on the train. Due to the length contraction of the train, we know that the marks on the train will be a distance γL apart when viewed in the train's frame (since this distance is what is length-contracted down to the given distance L in the platform frame).

How would someone on the train quantitatively explain to you why the marks are γL apart, considering that the bombs are only a distance $\frac{L}{\gamma}$ apart in the train frame?

Solution

The resolution to the apparent paradox is that the explosions do not occur simultaneously in the train frame. As the platform rushes past the train, the rear bomb explodes before the front bomb explodes. The front bomb then gets to travel farther by the time it explodes and leaves its mark. The distance between the marks is therefore larger than you might naively expect. Let's be quantitative about this.

Let the two bombs contain clocks that read a time t when they explode (they are synchronized in the ground frame). Then in the frame of the train, the front bomb's clock reads only $t - \frac{L v}{c^2}$ when the rear bomb explodes when showing a time t. (This is the result we found in the section "The Head Start" in Lecture 1; the two bombs may be considered to be the ends of the train.) The front bomb's clock must therefore advance by a time of $\frac{L v}{c^2}$ before it explodes. But since the train sees the bombs' clocks running slow by a factor γ, we conclude that in the frame of the train, the front bomb explodes a time $\frac{\gamma L v}{c^2}$ after the rear bomb explodes. During this time of $\frac{\gamma L v}{c^2}$, the platform moves a distance $\frac{\gamma L v^2}{c^2}$ relative to the train.

Therefore, the total distance between the two bomb blasts will equal

$$\frac{L}{\gamma} + \frac{\gamma L v^2}{c^2} = \frac{L}{\gamma}\left(\frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right) = \frac{L \gamma^2}{\gamma} = L \gamma \tag{10}$$

as desired. □

Extra Problem: Time Dilation and $\frac{L v}{c^2}$

Complementary Section: Train and the Bomb (aka Ladder in Barn)

Example

A train and a tunnel both have proper lengths L. The train speeds toward the tunnel, with speed v. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, this sensor tells the bomb to disarm itself. Does the bomb explode?

Solution

Yes, the bomb explodes. This is clear in the frame of the train. In this frame, the train has length L, and the tunnel speeds past it. The tunnel is length-contracted down to $\frac{L}{\gamma}$. Therefore, the far end of the tunnel passes the front of the train before the near end passes the back, and so the bomb explodes.

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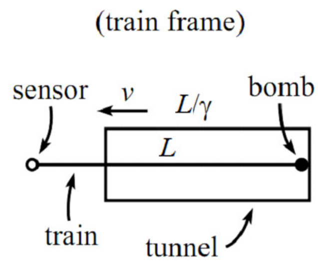

We may, however, look at things in the frame of the tunnel. Here the tunnel has length L, and the train is length-contracted down to $\frac{L}{\gamma}$.

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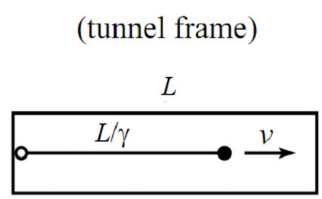

Therefore, the deactivation device gets triggered before the front of the train passes the far end of the tunnel, so you might think that the bomb does not explode. We appear to have a paradox.

The resolution to this paradox is that the deactivation device cannot instantaneously tell the bomb to deactivate itself. It takes a finite time for the signal to travel the length of the train from the sensor to the bomb. It turns out that this transmission time makes it impossible for the deactivation signal to get to the bomb before the bomb gets to the far end of the tunnel, no matter how fast the train is moving. Let's prove this.

The signal has the best chance of winning this "race" if it has speed c. So let us assume this is the case. If the signal gets to the front of the train before the front of the train reaches the front of the tunnel, the bomb will not explode. Assume for the sake of a contradiction that the bomb does not explode. Noting that the front of the train still moves while the signal propagates, this would happen if

$$\frac{L}{\gamma(c-v)} < \frac{1}{v}\left(L - \frac{L}{\gamma}\right) \tag{13}$$

which would imply the following inequalities (using the notation $\beta \equiv \frac{v}{c}$),

$$\frac{v}{c-v} < \gamma - 1 \tag{14}$$

$$\frac{1}{1-\beta} < \gamma \tag{15}$$

$$\frac{1}{1-\beta} < \frac{1}{\left(1-\beta^2\right)^{1/2}} \tag{16}$$

$$\left(1-\beta^2\right)^{1/2} < 1-\beta \tag{17}$$

$$(1+\beta)^{1/2} < (1-\beta)^{1/2} \tag{18}$$

This last inequality can never be true, a contradiction. Therefore, the signal always arrives too late, and the bomb always explodes. □

Advanced Section: The Expanded Train and a Bomb

Advanced Section: Rotated Square